

Optimal performance of heat engines with a finite source or sink and inequalities between means

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Given a system with a finite heat capacity and a heat reservoir, and two values of initial temperatures, T_+ and $T_- (< T_+)$, we enquire, in which case the optimal work extraction is larger: when the reservoir is an infinite source at T_+ and the system is a sink at T_- , or, when the reservoir is an infinite sink at T_- and the system acts as a source at T_+ ? It is found that in order to compare the total extracted work, and the corresponding efficiency in the two cases, we need to consider three regimes as suggested by an inequality, the so-called arithmetic mean-geometric mean inequality, involving the arithmetic and the geometric means of the two temperature values T_+ and T_- . In each of these regimes, the efficiency at total work obeys certain universal bounds, given only in terms of the ratio of initial temperatures. The general theoretical results are exemplified for thermodynamic systems for which internal energy and temperature are power laws of the entropy. The conclusions may serve as benchmarks in the design of heat engines, where we can choose the nature of the finite system, so as to tune the total extractable work and/or the corresponding efficiency.

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I. INTRODUCTION

Thermodynamics is regarded as a discipline with a formal simplicity, but still covering a wide domain of applicability. One of the central problems in thermodynamics is the extent of heat-to-work conversion, with its focus on maximal work or power output and the consequent efficiency of the process. The seminal results of Carnot apply to the case of infinite reservoirs. However, in recent years, the study of the role of finite reservoirs has also caught attention [1–7]. This is motivated by practical considerations such as a limited supply of fuel (a finite heat source), or the working medium being in contact with a small environment (sink) which may be the case in small-scale devices, or even relevant for the design of modern cities.

On the other hand, algebraic inequalities between the means hold a kind of poetic fascination. One of the most important [8] and well-known is the arithmetic mean-geometric mean (AM-GM) inequality, stated as follows. For two real positive numbers, a and b , with arithmetic mean $A(a, b) = (a + b)/2$ and geometric mean $G(a, b) = \sqrt{ab}$, we have

$$\frac{a + b}{2} \geq \sqrt{ab}, \quad (1)$$

with equality only if $a = b$. Such inequalities are useful in proving elementary results in many disciplines [9, 10]. Especially, in the context of macroscopic thermodynamics, the second law of increase of entropy may be argued as follows [11]. Consider n systems with a constant heat capacity C and initial temperatures, $\{T_i | i = 1, \dots, n\}$. Placed in mutual thermal contact, these systems come to equilibrium at a common final temperature, say T_f . From the energy conservation condition (the first law), we have $\sum_i C(T_i - T_f) = 0$, which implies $T_f = \sum_i T_i/n$. Now the total entropy change: $\Delta S = \sum_i \int_{T_i}^{T_f} (C/T) dT = nC(\ln T_f - \ln(\prod_i T_i)^{1/n})$, so by virtue of the AM-GM inequality [12], we get $\Delta S \geq 0$ [13–16]. Thus in the above argument, the manifestation of AM-GM inequality is specifically tied to the assumption of a particular model system. By assuming systems other than perfect gases, one can invoke inequalities between other means.

It is apparent that alternative thermodynamic processes, such as optimal work-extracting processes, would exhibit a similar connection between physical models and specific inequalities between the means. In this paper, our objective is to compare the work output capacity and efficiency of two complementary scenarios, involving a finite system and a reservoir. During this analysis, we will uncover a rather general role of the AM-GM inequality. In

particular, we will address the following question. Assume a pair of values for temperature, say T_+ and $T_- (< T_+)$, and a system A with a finite heat capacity. Also, a heat reservoir is present such that if the system is at temperature T_+ , the reservoir is a sink at T_- . Conversely, if the system is at T_- , then the reservoir is a hot source at T_+ . Which of these two situations (see Fig. 1) would yield a larger amount of extractable work, due to temperature difference? We answer this question by assuming that the process of maximal work extraction is carried out by some working medium (whose details are not important) via infinitesimal reversible heat cycles between system A and the reservoir.

In practical terms, we may consider a toy engine which can ideally work in a reversible manner, utilizing the temperature gradient between system A and the environment. Let T_+ and T_- be the environment temperatures, say, in summer and in winter season, respectively. So in summer, we cool the system A to temperature T_- , while in winter, we have to heat up the system to temperature T_+ , in order to run the engine. The engine works till it equilibrates at the specific temperature of the environment. When will the engine yield a larger amount of total work, in summer, or in winter?

The paper is organized as follows. In Section II, we describe the framework using two scenarios for work extraction due to temperature difference between a finite system and a heat reservoir. In Subsection II.A, the total extracted work and the corresponding efficiency are compared for the two scenarios. In Section III, physical examples are given based on thermodynamic systems where the temperature and the internal energy are related to the entropy by power laws. Section IV discusses the bounds on the efficiency at total work. Finally, Section V is devoted to summary and concluding remarks.

II. WORK FROM A FINITE SYSTEM AND A RESERVOIR

To set up the thermodynamic framework, consider system A following a certain fundamental relation $U = U(S, V, N)$. It has equilibrium states described by energy U_+ , entropy S_+ at temperature T_+ , and alternatively, by U_- and S_- at T_- , with some fixed values of volume V and number of moles N . For simplicity, we consider only systems with a positive heat capacity ($C_V > 0$). This implies that $U_+ > U_-$ and $S_+ > S_-$.

Now, we first assume that system A acts as a finite heat sink at temperature T_- , relative to a very large hot reservoir (source) at temperature T_+ . We couple the two by running

infinitesimal heat cycles, which successively increase the temperature of A, till A comes in equilibrium with the hot source, see Fig.1 (i). At an arbitrary intermediate stage, when the temperature of A is T , the small amount of heat removed from the source dQ_h is converted into an amount of work dW with maximal (Carnot) efficiency $\eta = 1 - T/T_+$. The heat discarded to the sink is $dQ_c = C_V dT$. Then, we can write $dW = \eta(1 - \eta)^{-1} dQ_c$. The total extracted work is given by:

$$W_+ = \int_{T_-}^{T_+} dW \quad (2)$$

$$= \int_{T_-}^{T_+} \frac{\eta}{1 - \eta} C_V dT \quad (3)$$

$$= T_+(S_+ - S_-) - (U_+ - U_-). \quad (4)$$

The heat absorbed from the hot source is $Q_+ = T_+(S_+ - S_-)$. Then the efficiency at total work, $\eta_+ = W_+/Q_+$, is calculated to be:

$$\eta_+ = 1 - \frac{1}{T_+} \frac{U_+ - U_-}{S_+ - S_-}. \quad (5)$$

Then, we consider the alternative situation in which A acts as a finite source at temperature T_+ , relative to an infinite sink at T_- , see Fig.1 (ii). Again, we extract the maximal work by utilizing the temperature gradient between A and the reservoir, till A is at temperature T_- . Then, after a similar calculation [5] as above, the total work obtained is

$$W_- = (U_+ - U_-) - T_-(S_+ - S_-). \quad (6)$$

This is termed as exergy in the engineering literature [17]. The heat absorbed from the source is $Q_- = U_+ - U_-$, while the efficiency of the process $\eta_- = W_-/Q_-$ is given by

$$\eta_- = 1 - T_- \frac{S_+ - S_-}{U_+ - U_-}. \quad (7)$$

Thus for the toy engine mentioned in Introduction, W_+ and η_+ (W_- and η_-) may refer to the total work and the corresponding efficiency in summer (winter) season.

A. The Comparison

Now we compare the amounts of extracted work, and the efficiencies, in these alternative set-ups. For that purpose, we recall the classic result in calculus, known as the *mean value*

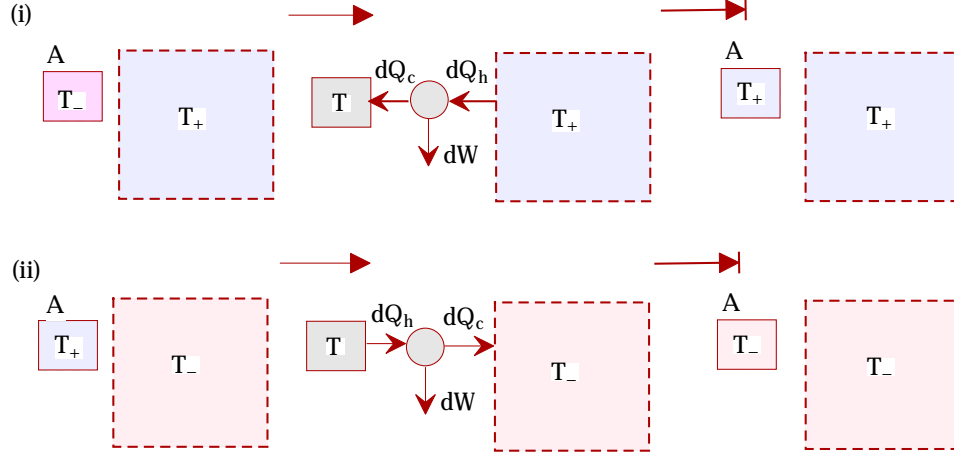


FIG. 1. Schematic of the reversible heat engine between a finite system and a heat reservoir, for a given pair of initial temperatures (T_+, T_-) : (i) System A is a finite sink at T_- and is coupled to an infinite source at T_+ , via heat engine. Work extraction W_+ , Eq. (4), is completed when the temperature of A becomes T_+ . (ii) System A is a finite source at T_+ and is coupled to an infinite sink at T_- , via heat engine. Total extracted work is W_- , Eq. (6), when the temperature of A becomes T_- .

theorem. Consider a continuous and differentiable function $U(S)$ in the domain $[S_-, S_+]$, with the derivative $T(S) = dU/dS$. Let us denote: $U(S_{\pm}) = U_{\pm}$. Following the theorem, there is a point S_m strictly within this interval ($S_+ > S_m > S_-$), at which the derivative of the function U , i.e. $T(S_m) \equiv T_m$, is given by:

$$T_m = \frac{U_+ - U_-}{S_+ - S_-}. \quad (8)$$

We also assume $T(S)$ to be monotonic increasing function, or, in other words, $U(S)$ is a convex function. In the context of thermodynamics, this assumption implies positive heat capacity (C_V) of the system. Then it follows that $T(S_+) > T(S_m) > T(S_-)$, or alternatively, $T_+ > T_m > T_-$.

Now, depending on the nature of the thermodynamic system i.e. the form of the function $U(S)$, T_m can take values relative to $A(T_+, T_-)$ and $G(T_+, T_-)$, such that we have the

following situations:

$$\begin{aligned}
 (a) \quad & T_+ > T_m \geq \frac{T_+ + T_-}{2} > \sqrt{T_+ T_-} > T_- \\
 (b) \quad & T_+ > \frac{T_+ + T_-}{2} > T_m > \sqrt{T_+ T_-} > T_- \\
 (c) \quad & T_+ > \frac{T_+ + T_-}{2} > \sqrt{T_+ T_-} \geq T_m > T_-
 \end{aligned} \tag{9}$$

We choose the means A and G to split the interval (T_-, T_+) into three regions, because for $T_m = (T_+ + T_-)/2$, we have $W_+ = W_-$, and for $T_m = \sqrt{T_+ T_-}$, we have $\eta_+ = \eta_-$. This helps naturally to compare the magnitudes of work, and efficiency. Thus, if case (a) holds, then applying $T_m \geq (T_+ + T_-)/2$, and using Eqs. (8), (4) and (6), we obtain $W_+ \leq W_-$. In this case, due to AM-GM inequality, we also have $T_m > \sqrt{T_+ T_-}$, which implies $\eta_+ < \eta_-$, due to Eqs. (8), (5) and (7).

Similarly, if case (b) applies, then we conclude that $W_+ > W_-$, but due to AM-GM inequality, we have $\eta_+ < \eta_-$. If case (c) is true, i.e. $\sqrt{T_+ T_-} \geq T_m$, it implies $\eta_+ \geq \eta_-$. Further, due to $(T_+ + T_-)/2 > T_m$, we also have $W_+ > W_-$. The above three scenarios are summarized in Table I.

Thus we see that the comparison of T_m with $A(T_+, T_-)$ decides the relative magnitudes of W_+ and W_- , whereas the comparison of T_m with $G(T_+, T_-)$, serves to compare η_+ and η_- . In these comparisons, the AM-GM inequality provides a sort of background against which T_m takes values depending on the nature of system A (see examples below). In terms of practical utility, the goal behind modelling of heat engines is to characterize their optimal working regimes. In this regard, if we are given a finite system A and a constraint to run the engine in one of the two scenarios, denoted as (i) and (ii) in the above, then a particular choice can be motivated as follows. In case the system A falls in category (a) of Table I, then choice (ii) provides a higher total work output and a higher efficiency. On the other hand, if system A belongs to category (c), then the choice (i) would provide a higher work output and a higher efficiency. In case the system belongs to regime (b), we have a situation with a trade-off. If we opt for a higher work output then the efficiency obtained is less, and vice versa. Heuristically, one may be able to make a choice in this situation as follows. A focus on a higher efficiency may become important, if the substance (system A) is in short supply or if the economic/ecological costs of preparing the system, in the desired state, are

(a)	(b)	(c)
$W_+ \leq W_-$	$W_+ > W_-$	$W_+ > W_-$
$\eta_+ < \eta_-$	$\eta_+ < \eta_-$	$\eta_+ \geq \eta_-$

TABLE I. Comparison of total work, Eqs. (4) and (6), and efficiency at total work, Eqs. (5) and (7), corresponding to regimes (a), (b) and (c) in Eq. (9).

rather high. On the other hand, if such costs are not a consideration, then one may focus on higher total work, with the corresponding efficiency being less of a concern.

III. EXAMPLES

In this section, we illustrate the various cases noted in the above, by taking examples from different types of physical systems. Consider a class of thermodynamic systems that obey: $U \propto S^\omega$ and $T \propto S^{\omega-1}$, where ω is a constant real number. For heat capacity to be positive, we must have $\omega > 1$. So, T_m is evaluated to be:

$$T_m = \frac{1}{\omega} \frac{T_+^{\omega/(\omega-1)} - T_-^{\omega/(\omega-1)}}{T_+^{1/(\omega-1)} - T_-^{1/(\omega-1)}}. \quad (10)$$

It is convenient to introduce the generalized mean [18, 19] of two real, positive numbers (a, b):

$$E_r(a, b) = \frac{r-1}{r} \frac{a^r - b^r}{a^{r-1} - b^{r-1}}. \quad (11)$$

In our case, $T_m = E_r(T_+, T_-)$ with $r = \omega/(\omega-1)$. For $r = 2$ ($\omega = 2$), $E_2(T_+, T_-) = (T_+ + T_-)/2$. For $r = 1/2$ ($\omega = -1$), $E_{1/2}(T_+, T_-) = \sqrt{T_+ T_-}$. Since $E_r(a, b)$ is increasing in parameter r [20], it follows that, for $r \geq 2$ or $\omega \geq 2$, we have $T_m = E_r(T_+, T_-) \geq E_2(T_+, T_-)$, which implies $T_m \geq (T_+ + T_-)/2$, or case (a). Therefore, for $2 > \omega > 1$, the system corresponds to case (b).

Some examples of physical systems in the above class, for appropriate values of T_+ and T_- , are: $\omega = 4/3$ (black-body radiation), $\omega = 5/3$ (degenerate Bose gas) and $\omega = 2$ (ideal Fermi gas). The case of a perfect-gas system, can be discussed as the limit $r \rightarrow 1$, which yields $E_1(T_+, T_-) = L(T_+, T_-)$, known as the logarithmic mean [21, 22]:

$$L(T_+, T_-) = \frac{T_+ - T_-}{\ln T_+ - \ln T_-}. \quad (12)$$

Logarithmic mean temperature difference is a useful measure of the effectiveness with which a heat exchanger can transfer heat energy [23]. This mean satisfies:

$$\frac{T_+ + T_-}{2} > L(T_+, T_-) > \sqrt{T_+ T_-}. \quad (13)$$

So if $T_m = L(T_+, T_-)$, then due to the above inequality, we have an instance of case (b). Thus with a perfect-gas system, the finite-sink/infinite-source setup produces more work than finite-source/infinite-sink setup ($W_+ > W_-$), although the efficiency at total work follows the reverse order ($\eta_+ < \eta_-$).

As our final model system, let A consist of N non-interacting, localized spin-1/2 particles [24]. Each particle can be regarded as a two-level system, with energy levels $(0, \epsilon)$. The mean energy for this system, in the limit of high temperatures such that $\epsilon \ll kT$, on keeping terms only upto $(\epsilon/kT)^2$, can be approximated as: $U \approx N(\epsilon/2 - \epsilon^2/4kT)$, with entropy $S \approx Nk(\ln 2 - \epsilon^2/8k^2T^2)$. Then from Eq. (8), we have: $T_m = 2T_+T_-/(T_+ + T_-)$, which is the well-known harmonic mean $H(T_+, T_-)$. This mean is strictly less than $G(T_+, T_-)$, and thus our spins-system lies in regime (c).

IV. BOUNDS ON EFFICIENCY

So far, we have noted the comparison between work characteristics for the two given scenarios. In the following, we point out that within a given scenario, the efficiency at total extracted work obeys definite bounds, which are specific to each of the regimes (a), (b) and (c). Thus if $T_m \geq (T_+ + T_-)/2$, then we get from Eq. (5), $\eta_+ \leq \eta_C/2$ where $\eta_C = 1 - T_-/T_+$ is the Carnot limit. Also from Eq. (5), we get $\eta_- \geq \eta_C/(2 - \eta_C)$. Similarly, in regime (c), when $T_m \leq \sqrt{T_+ T_-}$, we get $\eta_+ \geq \eta_{CA}$ and $\eta_- \leq \eta_{CA}$, where $\eta_{CA} = 1 - \sqrt{T_-/T_+}$ [25, 26], which is popularly known as CA-efficiency, after F. L. Curzon and B. Ahlborn who rediscovered this formula [27], see also [28]. These comparative bounds are summarized in Table II, as well as they are depicted in Figs. 2 and 3. Note that the efficiencies $\eta_C/2$, η_{CA} and $\eta_C/(2 - \eta_C)$ are frequently discussed in the context of maximum power output in finite-time models [25–27, 29, 30]. But we observe that, here, within a quasi-static framework, η_{CA} serves to separate η_+ and η_- in regimes (b) and (c).

The above bounds are universal as they depend only on the ratio of the initial temperatures. Note that the actual expressions, (5) and (7), do depend, in general, on the nature

(a)	(b)	(c)
$0 < \eta_+ \leq \eta_C/2$	$\eta_C/2 < \eta_+ < \eta_{CA}$	$\eta_{CA} \leq \eta_+ < \eta_C$
$\frac{\eta_C}{2-\eta_C} \leq \eta_- < \eta_C$	$\eta_{CA} < \eta_- < \frac{\eta_C}{2-\eta_C}$	$0 < \eta_- \leq \eta_{CA}$

TABLE II. The bounds obeyed by efficiencies at total extracted work, η_+ and η_- , in respective regimes given in Eq. (9), where $\eta_C = 1 - T_-/T_+$ and $\eta_{CA} = 1 - \sqrt{T_-/T_+}$.

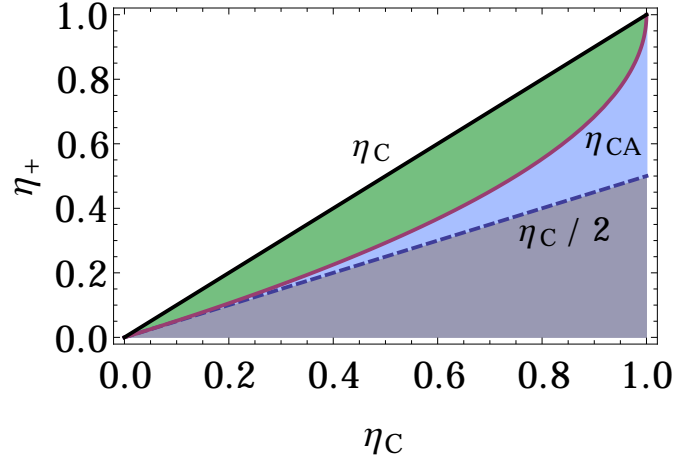


FIG. 2. Bounds on efficiency η_+ , in the regimes (from bottom to top) (a), (b), and (c), as given in Table II.

of system A. But close to equilibrium, even the general expressions for η_+ and η_- exhibit a universality. Thus assuming linear response, we can expand energy upto second order in the entropy difference $\delta S = S_+ - S_-$ [7]:

$$U(S_-) = U(S_+) - T_+ \delta S + \frac{1}{2} \left. \frac{dT}{dS} \right|_{S=S_+} (\delta S)^2. \quad (14)$$

Using the above expansion in Eq. (8), and upon simplifying, we get $T_m = (T_+ + T_-)/2$. This implies that $W_+ = (T_+ - T_-)\delta S/2 = W_-$. Thus, under linear response, the extracted work is same in both the cases. However, the efficiency at total work is approximated as $\eta_+ = \eta_C/2$ and $\eta_- = \eta_C/(2 - \eta_C)$. These expressions are consistent with the findings of Ref. [7], where the lower and the upper bounds for efficiency with unequal-sized source and sink, obey the same expressions.

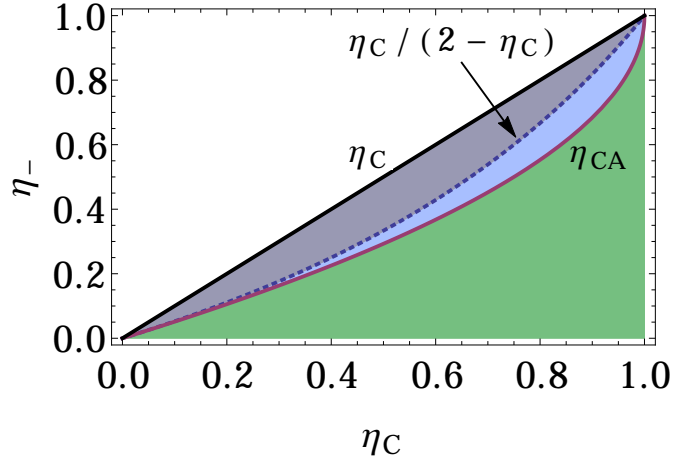


FIG. 3. Bounds on efficiency η_- , in the regimes (from top to bottom) (a), (b), and (c), as in Table II.

V. CONCLUDING REMARKS

We close this investigation by making a few remarks. Apart from an entropy-conserving process, we may analyze an energy-conserving process. The initial and final situations are the same as (i) and (ii) in Fig.1. Specifically, for situation (i), an amount of heat energy $U_+ - U_-$ is removed quasi-statically from the reservoir and deposited in the same manner with the cold system. The change in entropy of system A is $(S_+ - S_-) > 0$. The change in entropy of the reservoir is: $-(U_+ - U_-)/T_+$. Thus the total change in the entropy of the universe is:

$$\Delta S_+ = (S_+ - S_-) - \frac{U_+ - U_-}{T_+}. \quad (15)$$

Similarly, if we consider situation (ii), we can conclude that the total entropy change of the universe, in an energy-conserving process, would be:

$$\Delta S_- = -(S_+ - S_-) + \frac{U_+ - U_-}{T_-}. \quad (16)$$

Now, if we wish to compare the entropy production in the above two cases, then we are led to consider the following situations:

$$\begin{aligned} (a') \quad & T_+ > T_m \geq \frac{2T_+T_-}{T_+ + T_-} > T_- \\ (b') \quad & T_+ > \frac{2T_+T_-}{T_+ + T_-} > T_m > T_-. \end{aligned} \quad (17)$$

It is easy to see that if case (a') is true, then $\Delta S_- \geq \Delta S_+$. The inverse inequality is valid, if case (b') holds. Thus for an energy-conserving process, we see that the inequality between generalized mean T_m , and $H(T_+, T_-)$, quantifies the relative magnitudes of ΔS_- and ΔS_+ .

Finally, we consider an interesting meaning of T_m , given by Eq. (8), in the sense of an effective temperature. Take two heat reservoirs with temperatures T_m and $T_- (< T_m)$. Let $Q_m = U_+ - U_-$, be the heat extracted by the working medium from the hot reservoir in a reversible cycle. Here U_{\pm} refer to the energies of the working medium. Then the change in entropy of the hot reservoir is $T_m Q_m = S_+ - S_-$. The total extractable work in a reversible cycle is then $(T_m - T_-)(S_+ - S_-)$, which is the same as W_+ in Eq. (6). The Carnot efficiency of this process is $\eta_m = 1 - T_-/T_m$, which is Eq. (7). A similar conclusion follows for the other scenario, when we consider two heat reservoirs at temperatures T_+ and $T_m (< T_+)$. Thus T_m serves as the effective temperature of one of the two heat reservoirs in an equivalent reversible cycle, which extracts the same amount of work and with the same (Carnot) efficiency.

Concluding, the main focus of this paper was the comparison of performance of a reversible heat engine operating between a finite system and an infinite reservoir, by switching the role of the source and the sink. We compared the total extracted work in the two cases, and the corresponding efficiency of the engine at those values of the work. Interestingly, we find that the conditions for comparison are determined by basic mathematical inequalities between the means, in particular the AM-GM inequality. The present instance of this inequality does not depend specifically on the nature of the system as was the case in earlier studies. The efficiency at total work is naturally split into three regimes, based on this inequality. The bounds separating these regimes are variously given as $\eta_C/2$, η_{CA} and $\eta_C/(2 - \eta_C)$. This highlights a new significance of these expressions for efficiency, which are usually discussed in regard to power output optimization in finite-time models. The utility of our conclusions may also be discussed in the context of the toy engine mentioned in the Introduction. Thus, for a given pair of temperatures (T_+, T_-) , we can characterize system A, or our device, based on the regime (a) , (b) or (c) , to which it corresponds. This determines how W_+ and W_- compare with each other, which further guides whether η_+ will be greater, or lesser, relative to η_- . Moreover, in a particular regime, we know from Table II, the bounds within which the efficiency at total work is located. Thus given a choice of system A, the efficiency at total work is restricted within a certain range. Although derived for quasi-static processes, these bounds may serve as benchmarks for tuning the performance

of real devices, and can be a useful element in their design.

One of the limitations of our analysis may be that we have considered idealized quasi-static processes. In practical cases, the engines and other thermodynamic machines work in finite cycle-times. Thus an extension of our analysis within an irreversible framework [5] may help to see how the above conclusions are retained or modified in finite-time models, at least under linear response or beyond that [7]. Another interesting line of enquiry seems to be the connection of the bounds on efficiency with the principles of inductive inference [31, 32]. Finally, it is hard to ignore the aesthetic motivation in revealing other inequalities, possibly new, with these investigations. But, this is left for future work.

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